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The Reality of Resonances in the Solar System

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The object of this paper is to attempt a quantitative evaluation of the probability of a given resonant structure. It is shown that the formation of a "good" resonant structure by chance is not very likely, and that the random probability of the resonant structure of the solar system is less than 10^{-10}.

A theory of the resonant structure of the solar system by the present author (Molchanov, 1968) follows from an earlier, more general argument (Molchanov, 1966) that oscillating systems which have attained evolutionary maturity are inevitably resonant, and that their structure is given by sets of integers, just as in quantum systems. This raises interesting questions about the possible structure of planetary systems.

However, the resonance relations are not satisfied exactly; there is always some deviation. Because any real numbers can be approximated by rational numbers with arbitrary accuracy an important question remains unanswered: What is the significance of a statement about resonance?

I. Unperturbed Systems

Strictly speaking the problem of resonance should be formulated in the following manner: We are given (see Molchanov, 1968) a multiple oscillating system containing the small parameter ϵ:

$$d\boldsymbol{\phi}/dt = \boldsymbol{\omega}(\mathfrak{J}) + \epsilon\boldsymbol{\Omega}(\mathfrak{J}, \boldsymbol{\phi}, \epsilon)$$
$$d\mathfrak{J}/dt = \epsilon\mathscr{F}(\mathfrak{J}, \boldsymbol{\phi}, \epsilon), \qquad (1)$$

where

$$\boldsymbol{\phi} = \{\phi_i\}$$

is the phase vector, and

$$\mathfrak{J} = \{\mathfrak{J}_k\}$$

is the set of first integrals of the unperturbed system for $\epsilon = 0$. System (I) is periodic with period 2π for each of the phases.

It would be necessary to find resonant solutions of System (I) and to show that the observed data correspond to the solution with an accuracy acceptable in celestial mechanical investigations. However, at present this problem has not even been stated correctly, and its solution will probably not be possible for a long time.

In general only the unperturbed system is being studied,

$$d\boldsymbol{\phi}/dt = \boldsymbol{\omega}(\mathfrak{J})$$
$$d\mathfrak{J}/dt = 0 \qquad (2)$$

and so the studies are of a heuristic and probabilistic nature. Nevertheless, it is necessary to retain one important requirement which arises from the nature of a complete system—only those transformations of phase variables which retain periodicity are allowed. It can be verified that this is equivalent to the requirement that the change of variables is of the form

$$\boldsymbol{\Psi} = A\boldsymbol{\phi}, \qquad (3)$$

where A is a matrix with integer elements which is unimodular, i.e., its determinant is equal to unity.

The state of an unperturbed system is completely defined by the frequency vector ω transformed by the formula

$$\boldsymbol{\nu} = A\boldsymbol{\omega}. \qquad (4)$$

The introduction of these transformations enables all maximum resonant systems to be determined quite simply. Examine the vector $\boldsymbol{\nu}$ with only one[1] nonzero component

$$\nu_1 = 0, \quad \nu_2 = 0, \quad \ldots, \quad \nu_n = \nu \neq 0. \tag{5}$$

We consider the arbitrary unimodular matrix A and construct the vector $\boldsymbol{\omega}$ with the help of the inverse matrix A^{-1} (which will also have integer elements since the determinant of $A = 1$):

$$\boldsymbol{\omega} = A^{-1}\boldsymbol{\nu}. \tag{6}$$

From Eqs. (4) and (5) it is clear that, except for the last row, the matrix A is composed of coefficients of resonance relations. And Eq. (6) shows that all frequencies ω_i are essentially integral multiples of ν.

Thus each such unimodular matrix A and number $\boldsymbol{\nu}$ produce a resonance vector. Conversely, as shown in (1) any resonance vector $\boldsymbol{\omega}$ can be represented as in Eq. (6).

II. Statement of the Problem

Such a description facilitates a quantitative formulation of the reality of a suspected resonant system. The basic object is clear—if systems similar to the solar system are extremely rare, they cannot be a result of chance. Then the resonance structure requires an explanation in terms of evolution.

However, the words "rare" and "similar" should first be defined precisely. Rarity is most commonly explained in the context of the theory of measurement as belonging to a set of small measure. But it is inconvenient to consider phase volume as a measure, since the resonance of a system is invariant to scale transformation and the volume of the whole space is infinite. Therefore "relative" measure will be used,

$$d\mu = \prod_i{}' \frac{d\omega i}{\omega i} \tag{7}$$

in addition to the finite parallelepipeds in frequency space. It is more accurate to consider the space of relative frequencies (as one of the frequencies can be assumed as unity) whose dimension is less than the number of frequencies by one and is equal to the number of resonance relations. The use of the quantity

$$\Delta\mu = \prod_i{}' \frac{\Delta\omega i}{\omega i} \tag{8}$$

is more valid to measure the closeness of the real vector to the theoretical (exactly resonant) vector. The primes on the product signs mean omission of the frequency which has been chosen as unity.

It is even more important to define what is meant by "similar to the solar system." Maximum resonant systems form a dense set everywhere; by "smearing" each point of this denumerable set by the parallelepiped (8) we shall obtain the whole of phase space.

However, the four matrices which describe the structure of the planetary and satellite systems of Jupiter, Saturn, and Uranus possess important distinguishing features which can be described intuitively as nearly triangular with not very large coefficients. This makes it possible to state the heuristic and undoubtedly controversial hypothesis that they are part of a restricted class of "good" resonant systems. The exact meaning of the word in quotes is given below for each of the systems. The general method for calculating the rarity P of a given system is also given below.

The real system being studied and the ideal theoretical system which approximates it generate a parallelepiped in a space with the logarithms of frequencies as coordinates if the former is taken as the "corner" and the latter as the center of the parallelepiped. All the remaining equally "good" resonant systems are also enclosed in this neighborhood. A set of systems not "worse" than the given system is obtained and its volume is calculated.

In the same space the "enveloping cube" is constructed in which each of the sides is determined by

$$a = \ln \omega_{max} - \ln \omega_{min}. \tag{9}$$

Its volume is simply a^{n-1}.

[1] In a similar manner systems in which the number of resonances is less than the number of phases by two can also be studied. In this case the vector $\boldsymbol{\nu}$ has two incommensurable nonzero components.

The ratio P of these volumes is also taken to be the quantitative measure of the rarity of the system being studied. As the neighborhoods of ideal points can intersect and the points themselves can be situated within the enveloping cube,

$$P < N(2^{n-1}\Delta\mu/a^{n-1}), \qquad (10)$$

where $\Delta\mu$ is the magnitude given by Eq. (8); the factor 2^{n-1} comes from the equality of clockwise and anticlockwise displacements; and N is the number of matrices A equally "good" as the one under study.

III. "Good" Matrices

The structural matrix of the planetary system (Molchanov, 1966) contains the classical resonance 2:5 of Jupiter and Saturn's periods and appears in the following form:

$$A_{cl} = \left\| \begin{matrix} 1 & -1 & -2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -6 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & +2 & -5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 5 & 1 \end{matrix} \right\| . \qquad (11)$$

However, besides the resonance relation

$$2\omega_{♃} - 5\omega_{♄} \approx 0 \qquad (12)$$

it is possible to denote another

$$\omega_{♃} - 2\omega_{♄} - \omega_{♅} - \omega_{♇} \simeq 0 \qquad (13)$$

which is satisfied somewhat more accurately than the classical relation. The discrepancy of the classical relation is 0.0135 and that of the new relation 0.0059. However, it is more correct to compare the relative errors, which are 0.0067 and 0.0059, respectively.

Changing the fifth row with coefficients of relation (13)—the second with the difference of the second and third, and the sixth with a linear combination of the four last resonance relations—it is possible to construct another structural matrix of the planetary system,

$$A_{new} = \left\| \begin{matrix} 1 & -1 & -2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -6 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{matrix} \right\| . \qquad (14)$$

The theoretical frequency vectors generated by these two matrices approximate the real vector equally well, as Table I shows. Therefore the theoretical possibility of intersection of neighborhoods of ideal points is realized even for planetary systems. It has already been shown that the question that arises, as to which of the matrices is the correct one, does not make sense in the framework of unperturbed equations; it requires the study of a complete system. Most of all, both variants are bad because they do not take into account the structural hierarchy of planetary systems. Our solar system comprises at least two groups: Mars is the last of the interior group of rocky planets; it is separated from Jupiter, which in turn is the first of the exterior group of gaseous planets, by a frequency interval greater than two "octaves."

Both variants of the structural matrices of planetary systems have pros and cons. The first matrix reduces to a more simple rational approximation for frequencies, whereas the second is considerably closer to structural matrices of satellite systems when one considers the most important properties of the second matrix. The latter feature is useful because it enables a sufficiently general definition of the class of systems with "good" resonances in each of the four cases. It seems inevitable that the principles of planetary and satellite formation will be defined more precisely in the future. This will result in the class of plausible "equally good" systems being reduced, and therefore will

TABLE I

FREQUENCIES OF THE PLANETARY SYSTEM

Planet	ω_{obs}	ω_{cl}	$\Delta\omega/\omega$	ω_{new}	$\Delta\omega/\omega$
Mercury	49.2508	49.2000	0.0010	49.1904	0.0012
Venus	19.2816	19.2571	0.0013	19.2619	0.0010
Earth	11.8618	11.8286	0.0028	11.8333	0.0024
Mars	6.3067	6.2857	0.0033	6.2857	0.0033
Jupiter	1.0000	1.0000	—	1.0000	—
Saturn	0.40269	0.4000	0.0069	0.40476	−0.0051
Uranus	0.141191	0.142857	−0.0118	0.142857	−0.0118
Neptune	0.071984	0.071428	0.0077	0.071428	+0.0077
Pluto	0.047499	0.047619	−0.0025	0.047619	−0.0025

only strengthen the argument developed here.

Comparison of the structural matrices of satellite systems

$$A_{\text{♄}} = \begin{Vmatrix} 1 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -2 & -1 & 0 & -2 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 4 \end{Vmatrix} \tag{15}$$

$$A_{\text{♅}} = \begin{Vmatrix} 1 & -1 & -1 & 0 & -1 \\ 0 & 1 & -1 & -2 & 1 \\ 0 & 0 & -2 & 1 & 5 \\ 0 & 0 & 1 & -6 & 6 \\ 0 & 0 & -2 & 3 & 2 \end{Vmatrix} \tag{16}$$

$$A_{\text{♃}} = \begin{Vmatrix} 1 & -2 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & -3 & 7 \\ 0 & 0 & -1 & 2 \end{Vmatrix} \tag{17}$$

reveals that they are almost triangular. Because of this property A_{new} is preferred over A_{kl} and this property serves as the basic definition of the class of "good" matrices.

The four cases of A can be represented as differences of two matrices—the "skeletal" part S and the "triangular" part T

$$A = S - T. \tag{18}$$

The matrix S in block form is given by

$$S = \begin{Vmatrix} E & 0 \\ 0 & \sigma \end{Vmatrix}, \tag{19}$$

where σ is a third order unimodular matrix (second order for Jupiter's satellites) and E is the unit matrix in the space complimentary to σ. Matrix T has zeros on its main diagonal (where matrix S has unities), everywhere below the diagonal and in positions occupied by matrix σ. In other words wherever S is nonzero T is zero and vice versa. In this sense A is not simply a difference but a superposition of matrices S and $-T$. Below the diagonal, matrix A cannot have more than three nonzero elements entering the matrix σ; hence the name "nearly triangular" is given to such matrices.

Unfortunately, the second and important property of "good" matrices, namely possessing "not very large" coefficients, cannot be stated in a sufficiently general form. But it can be described as follows: Basically the triangular parts are composed of zeros and unities and contain a small number of twos.[2] Threes, fours, fives, and sevens (unique in the whole solar system) are found only in "heads" σ of the skeletal parts of the structural matrix. Apart from this, nonzero elements of the triangular parts gravitate to the main diagonal and tend to be positive. These properties are seen more clearly if the "heads" σ and the triangular parts are written separately omitting all the known

[2] The only exception—namely six—suggests that the principle behind the structure of planetary systems is not quite well understood. Somehow the hierarchy has to be taken into account, but at the present it is not evident how this can be accomplished.

zero elements (σ and T without a subscript relate to the planetary system):

$$\sigma = \left\| \begin{array}{rrr} 1 & -2 & 0 \\ 1 & 0 & -3 \\ 0 & 1 & -1 \end{array} \right\|; \quad \sigma_{\text{♄}} = \left\| \begin{array}{rrr} 3 & -4 & 0 \\ 1 & 0 & -5 \\ 0 & -1 & 4 \end{array} \right\|;$$

$$\sigma_{\text{♅}} = \left\| \begin{array}{rrr} -2 & 1 & 5 \\ 1 & -6 & 6 \\ -2 & 3 & 2 \end{array} \right\|; \quad \sigma_{\text{♃}} = \left\| \begin{array}{rr} -3 & 7 \\ -1 & 2 \end{array} \right\| \tag{20}$$

$$T = \left\| \begin{array}{rrrrrrrr} 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ & & 2 & -1 & 1 & -1 & 0 & 0 \\ & & & 6 & 0 & 2 & 0 & 0 \\ & & & & 2 & 1 & 0 & 1 \\ & & & & & 2 & 1 & 1 \end{array} \right\| \tag{21}$$

$$T_{\text{♄}} = \left\| \begin{array}{rrrrrrr} 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ & 0 & 2 & 0 & 0 & 0 & 0 \\ & & 0 & 2 & 1 & 0 & 2 \\ & & & 2 & -1 & 0 & -1 \\ & & & & 2 & 2 & 0 \end{array} \right\| \tag{22}$$

$$T_{\text{♅}} = \left\| \begin{array}{rrrr} 1 & 1 & 0 & 1 \\ & 1 & 2 & -1 \end{array} \right\| \tag{23}$$

$$T_{\text{♃}} = \left\| \begin{array}{rrr} 2 & 0 & 0 \\ & 2 & 0 \end{array} \right\|. \tag{24}$$

IV. The Number of "Good" Matrices

The fact that the anatomies of such structural matrices have so much in common justifies the definition of another set of systems in terms of T and σ. This set of systems is defined as being not "worse" than the given system. Any such definition must be based on the properties of matrices T and σ for the specific system being studied.

Planetary System

The conditions T must satisfy, according to the properties of matrices (21), are

(1) Beyond the three diagonals adjacent to the main diagonal there should not be more than three unities. The remaining 20 elements are zeros.

(2) There should not be more than two negative elements.

(3) There can only be one location with a number greater than 2 and such a number should be below 7.

The matrix σ can be completely defined by two rows, i.e., six elements. In the case of planetary systems a sufficiently restricted class is obtained from two conditions:

(a) There should be at least four positions with unities and zeros.

(b) No element should be greater than 6.

Although any "good" matrix A produces "good" matrices T and σ, the reverse is not always true. Ideal vectors with negative frequencies or frequencies from beyond the boundaries of the enveloping cube of the system can give a "good" matrix like A.

Therefore, the total number N of systems not "worse" than the given system is estimated (from above) by

$$N \leqslant N_T N_\sigma. \tag{25}$$

Turning to the calculation of N_T: On the three diagonals in matrix T one of the 18 positions is occupied by numbers between 3 and 7, and the remaining places contain 0's, 1's, or 2's. Therefore the total number of combinations in which vacant positions can be filled is given by

$$n_1 = \frac{18!}{1!\,17!} 5^1 3^{17} = 1.15 \times 10^{10} \tag{26}$$

The three unities in the upper corner of matrix T can occupy the 15 vacant positions in n_2 ways, where n_2 is given by

$$n_2 = \frac{15!}{3!\,12!} = 4.55 \times 10^2. \tag{27}$$

The two minus signs still have to be accounted for. There are 21 nonzero elements and so

$$n_3 = 21!/2!\,19! = 2.1 \times 10^2. \tag{28}$$

By multiplying these numbers we obtain the number of "good" matrices T:

$$N_T < n_1 n_2 n_3 = 1.1 \times 10^{15}. \tag{29}$$

When calculating the number of good matrices, σ, it is necessary to take into account the requirement that the frequencies must be positive. If the signs in the first two columns of matrix σ are changed independently, this is equivalent to changing the signs of the first two frequencies and will give positive frequencies in one case out of four. In all other respects the reasoning is the same.

Thus, in four positions we have zeros and unities, and in two positions we have numbers from -6 to -2 and from 2 to 6. This gives

$$N_\sigma < \frac{1}{4}\frac{6!}{4!\,2!}3^4\,10^2 = 3.05 \times 10^4. \quad (30)$$

So the total number of "good" matrices for planetary systems is extremely large:

$$N < 3.35 \times 10^{19}. \quad (31)$$

Saturn's Satellites

In matrix T two diagonals are characteristic and there are three nonzero elements in the corner; the number of minus signs is two. As none of the numbers is greater than 2, the same arguments that have been used before give

$$n_1 = 3^{10} = 5.9 \times 10^4; \quad (32)$$

further

$$n_2 = \frac{15!}{3!\,12!}2^3 = 3.64 \times 10^3 \quad (33)$$

where the factor 2^3 appears because unities or twos may be present in the corner of the matrix. The number n_3 is obtained by taking into account the signs

$$n_3 = \frac{13 \times 12}{2 \times 1} = 78. \quad (34)$$

Therefore

$$N_T < n_1 n_2 n_3 = 1.67 \times 10^{10}. \quad (35)$$

The distribution of positions is somewhat changed for matrix σ and we obtain the figure

$$N_\sigma < \frac{1}{4}\frac{6!}{3!\,3!} = 3^3 \times 10^3 = 1.35 \times 10^5. \quad (36)$$

Finally

$$N_{♄} < 2.25 \times 10^{15}. \quad (37)$$

Uranus' Satellites

Here, there are no characteristic diagonals in matrix T; so the two factors n_1 and n_2 are not separable. The only negative number is a 2, the others being zeros and unities. Therefore zeros and unities could occur in six out of seven places and 2 could occur in one place; a minus sign should occur somewhere. This gives

$$N_T < 7 \times 7 \times 2^6 = 3.14 \times 10^3. \quad (38)$$

The arrangement of matrix σ is the most complicated of all. Zeros, unities, and twos occur in three positions; together with possible sign changes this gives five combinations. Three places are taken by 3's, 4's, and 5's giving six combinations. Thus

$$N_\sigma = \frac{1}{4}\frac{6!}{3!\,3!}5^3\,6^3 = 1.35 \times 10^5. \quad (39)$$

Finally,

$$N_{♅} < 4.24 \times 10^8. \quad (40)$$

Jupiter's Satellites

There are five positions in T; each position could be occupied by one of five numbers—zeros, unities, and twos, each positive or negative. Hence,

$$N_T < 5^5 = 3.12 \times 10^3. \quad (41)$$

There are only two important positions in σ, but there is the number 7. Therefore

$$N_\sigma < 7^2 = 49. \quad (42)$$

This means that

$$N_{♃} < 1.52 \times 10^5. \quad (43)$$

V. The Rarity of "Good" Systems

From the evaluation in the previous section it is seen that the number of good systems is indeed very large. Nevertheless, the total volume occupied in the logarithm of frequency space by all the systems not "worse" than planetary systems forms a very small part of the enveloping cube. From Table I we find

$$\Delta\mu = 1.10 \times 10^{-20} \tag{44}$$

$$a = 6.96 \tag{45}$$

and substituting these numbers in Eq. (10) for P, we obtain

$$P \leqslant 1.7 \times 10^{-5}; \tag{46}$$

This estimate gives the rarity only of the planetary system. The four numbers which give the rarity of the planetary system and the three satellite systems are necessary to obtain the rarity of the whole system.

The values of $\Delta\mu$ and a are necessary for the calculation and they can be extracted from Table II of Molchanov (1968) on the theoretical and observed frequencies[3] of satellite systems.

TABLE II

VALUES FROM TABLE II OF MOLCHANOV (1968)

Saturn's Satellites
$\Delta\mu = 2.08 \times 10^{-17}$
$a = 4.43$
$(\frac{1}{2}a)^7 = 2.64 \times 10^2$
Uranus' Satellites
$\Delta\mu = 3.03 \times 10^{-10}$
$a = 2.31$
$(\frac{1}{2}a)^4 = 1.8$
Jupiter's Satellites
$\Delta\mu = 5.7 \times 10^{-8}$
$a = 2.24$
$(\frac{1}{2}a)^3 = 1.4$

The results of the calculations are as follows:

$$P_{\text{♄}} < 1.8 \times 10^{-4} \tag{47}$$

$$P_{\text{♅}} < 7.2 \times 10^{-2} \tag{48}$$

$$P_{\text{♃}} < 6.3 \times 10^{-3}. \tag{49}$$

[3] This is the frequency scale given by the frequency of the most massive body in any given system.

Therefore the rarity of the solar system is given by the very small number

$$P' < 1.4 \times 10^{-12}. \tag{50}$$

In order to evaluate the result obtained we observe that the number of stars in our galaxy is of the order of 10^{11}. Even if each star were to be provided with a planetary system, our galaxy would not be sufficient to obtain by chance even one system similar to the solar system. At the present time people do not venture to press their exclusiveness to such an extent. Rather, the opposite tendency can be observed.

The arguments presented herein do not constitute a strict proof. Only a formal theory of resonance states in a complete system could provide such a proof. Thus the applicability of the theory of perturbed Hamiltonian systems to the solar system is quite questionable. This theory which in fact comes from celestial mechanics and forms one of the most interesting branches of mathematics is based on the general state of the frequency vector of a system.

NOTE ADDED IN PROOF 29 MAY 1969

The author is indebted to S. F. Dermott (1969) who indicated an error in the value for the frequency of Miranda quoted by Molchanov (1968). The correct value, $\omega_{obs} = 6.157$ leads to a resonance vector $(1\ -1\ -1\ 0\ -1)$ in $A_{\text{♅}}$, instead of the incorrect $(1\ -1\ -1\ 0)$ and to the correct value of $\omega_{theor} = 6.181$.

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